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Finite-and Fixed-time Stabilization of the Inhomogeneous Parabolic PDE Systems with In-domain or Boundary Control

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Abstract— **This paper introduces innovative finite-time and fixed-time state feedback stabilization techniques for systems modeled by inhomogeneous parabolic partial differential equations, with a focus on control strategies implemented either within the domain or at the boundaries. Diverging from previous studies, this research specifically examines the influence of an inhomogeneous source term in the PDE domain, particularly in scenarios where the source term is independent of the system state, output, and boundary conditions. The essential role of inhomogeneous parabolic PDE systems in various applications is examined. Initially, we address the problem of finite-time stabilization for inhomogeneous parabolic PDEs by first designing an in-domain controller for systems that are controllable within the domain, followed by the design of a boundary controller for systems controllable at the boundaries. Subsequently, we tackle the issue of fixed-time stabilization for inhomogeneous parabolic PDEs, where the convergence time is independent of the system states. This involves first designing an in-domain controller for systems controlled within the domain, and then developing a boundary controller for systems controlled at the boundaries. To evaluate the stability of the closed-loop system, the Lyapunov technique is employed for analysis. Finally, simulations are conducted to demonstrate the effectiveness of the proposed methodologies.**

Index Terms— **Finite-time and fixed-time control, Inhomogeneous PDE stabilization, In-domain and boundary control.**

I. Introduction

With advancements in science and engineering, system models based on Partial Differential Equations (PDEs) have garnered significant attention from researchers. Nowadays, numerous systems are modeled using an accurate PDE, including battery models [1], [2], heat transfer [3], and multi-agent systems [4]. PDEs offer greater accuracy compared to ordinary differential equations (ODEs) due to their spatial dependence and infinite-dimensional nature [5], [6]. Consequently, PDE control approaches have attracted substantial interest. These approaches are categorized into two groups [5]: in-domain control, where the controller is actuated within the domain of the system, and boundary control, where the controller is actuated at the system's boundaries. Extensive research has been conducted in both fields, with numerous publications (See [7]–[10]) and books (See [5], [11]) available on the subject.

In general, control problems are categorized into two groups [12]: regulation, where the system states need to be maintained at a constant value, and tracking control, where the system states need to follow a time-varying signal. Most PDE-based tracking problems involve stabilizing an inhomogeneous PDE system with a source term causing inhomogeneity in the domain [13]. In addition to tracking problems, which constitute a large group of control issues, many systems are inherently inhomogeneous, such as heat transfer [3], battery models [14], and Plasma [15].

Although [15] tries to stabilize such a system with manipulating the q-profile safety factor, [16] investigates the control of inhomogeneous quasi-linear systems where the source term is dependent on the system state, [17] attempt to track the output only with in-domain control, [18] investigates the boundary control of inhomogeneous parabolic PDEs with a new backstepping method where the Kernel gain is non smooth where proofs are not provided, [19] try to linearize a drilling model, and all related works in [11] only have the inhomogeneous boundary conditions mostly for output tracking, but to date no significant efforts have been made to stabilize such systems with a state-feedback when the source term of the inhomogeneous system is fully independent of the system states and outputs. While controlling such systems is relatively straightforward when the control is applied within the domain alongside the source term [13], it becomes more challenging when the source term and the control are located in different regions, necessitating boundary control.

Finite and fixed-time stabilization methods are developed to achieve a known and finite convergence time. In finite-time stabilization, the convergence time depends on the system state. In contrast, fixed-time stabilization ensures that the convergence time is independent of the system state [20]. [7] and [21] investigate the stabilization of homogeneous PDEs where a Lyapunov-based boundary control approach has been used to stabilize the system at a predefined time.

- To the best of the authors' knowledge, this study introduces the finite-time and fixed-time state feedback stabilization techniques for systems modeled by inhomogeneous PDEs, with a focus on control strategies implemented either in the domain or at the boundaries.
- To the best of the author's knowledge, this is the first study to examine the boundary stabilization of general class of parabolic inhomogeneous PDEs and develops finite-and fixed-time designs.
- Compared with [13], [17] which have studied inhomogeneous PDE stabilization with in-domain control, this paper differentiates itself by investigating **boundary control** where the system is stabilized in a **finite-time**.
- Compared with [15], [16], [18] which special class of PDE systems are considered and they have tried to develop an exponential backstepping based control, this

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paper differentiates itself by designing **finite-and fixed-time** controllers for general class of **parabolic PDEs**.
- Compared with [7], [21] which developed finite-and fixed-time control designs for homogeneous PDEs, this paper investigates the **inhomogeneous PDEs** and considers in-domain control.

The paper is organized as follows: Section II defines the inhomogeneous parabolic PDEs considered in this study. In Section III, we develop finite-time in-domain and boundary controllers for the defined PDE systems. Section IV investigates the fixed-time stabilization of these systems within the domain and boundary. Simulations are presented in Section V. The work is concluded in Section VI.

II. Problem Statement

In this section we discuss the systems modeled by inhomogeneous PDEs and definitions.

A. Inhomogeneous PDE systems

Battery systems [14] and heat transfer processes [3], among others, are inherently modeled by inhomogeneous PDEs due to the presence of source terms causing inhomogeneity. In the case of battery models, the inhomogeneity source term can be described as thermal runaway, which causes severe damage to the system and must be controlled. Similarly, in heat transfer models, inhomogeneity may result from friction, leading to undesirable temperature divergence that requires management. Generally, a dimensionalized modeled heat transfer system is represented as

$$T_t(t,x) = \epsilon T_{xx}(t,x) + \lambda T(t,x) + S(t,x), \tag{1a}$$

$$T(t,0) = 0, \quad T(t,1) = 0, \tag{1b}$$

$$T(0,x) = T_0(x). \tag{1c}$$

where $(t,x) \in \mathbb{R}^+ \times \Omega$ with $\Omega = \{x \mid 0 \le x \le 1\}$, let $T(t,x) \in \mathbb{R}$ represents the temperature of the system. ϵ and λ are the thermal diffusivity and reactivity, respectively where ϵ is a positive constant. $S(t,x) \in \mathbb{R}$ is the inhomogeneity source term which is known, bounded and independent of system temperature. The control aim here is to set $T(t,x)$ to zero and stabilize the system dynamic.

B. Tracking Problem in PDE systems

Platoons [22], and multi-agent systems [6], [13], [23] are good examples of systems modeled by homogeneous parabolic systems where a main goal for such a systems is to track a time-varing signal to shape a specific formation and follow a path. A general homogeneous parabolic PDE based on [13] is in the following form

$$\theta_t(t,x) = a\theta_{xx}(t,x) + b\theta_x(t,x) + c\theta(t,x), \tag{2a}$$

$$\theta(t,0) = \bar{\theta}(t,0), \quad \theta(t,1) = \bar{\theta}(t,1), \tag{2b}$$

$$\theta(0,x) = \theta_0(x). \tag{2c}$$

In the context of $(t,x) \in \mathbb{R}^+ \times \Omega$, let $\theta(t,x) \in \mathbb{R}$ represent the system state. Here, a, b, and c denote the diffusivity, advectivity, and reactivity coefficients, respectively where the diffusivity coefficient is positive. Consider $\bar{\theta}(t,x)$ is the desired signal that the system state aims to track, such that $\theta(t,x) \to \bar{\theta}(t,x)$. Define the error signal as $\tilde{\theta}(t,x) := \theta(t,x) - \bar{\theta}(t,x)$. The control objective is to set $\tilde{\theta}(t,x)$ to zero and stabilize the error dynamics. Based on (2), the error dynamic transforms into an inhomogeneous PDE as

$$\tilde{\theta}_t(t,x) = a\tilde{\theta}_{xx}(t,x) + b\tilde{\theta}_x(t,x) + c\tilde{\theta}(t,x) + s(t,x), \tag{3a}$$

$$\tilde{\theta}(t,0) = 0, \quad \tilde{\theta}(t,1) = 0, \tag{3b}$$

$$\tilde{\theta}(0,x) = \tilde{\theta}_0(x), \tag{3c}$$

in which $s(t,x) = \bar{\theta}_t(t,x) - a\bar{\theta}_{xx}(t,x) - b\bar{\theta}_x(t,x) - c\bar{\theta}(t,x)$ is the inhomogeneity source term. Note that $s(t,x) \in \mathbb{R}$. Stabilization of such system with an in-domain controller or a boundary controller with a finite predefined system is the main focus of this paper.

Assumption 1: Throughout this paper, it has been assumed that the inhomogeneity source term in the system dynamics is known, bounded, and entirely independent of the system states, boundaries, and outputs.

III. Finite-Time Stabilization

In this section, we focus on systems modeled by inhomogeneous parabolic PDEs and aim to design controllers that achieve stabilization with finite convergence time. Initially, we develop an in-domain controller for the system as formulated in (4), ensuring finite-time stability. Subsequently, we design a boundary controller as formulated in (17), also achieving finite-time stabilization of the system.

A. In-domain Control

Consider an inhomogeneous PDE as

$$\begin{aligned}\tilde{\theta}_t(t,x) =& a\tilde{\theta}_{xx}(t,x) + b\tilde{\theta}_x(t,x) + c\tilde{\theta}(t,x) \\ &+ s(t,x) + u(t,x),\end{aligned} \tag{4a}$$

$$\tilde{\theta}(t,0) = 0, \quad \tilde{\theta}(t,1) = 0, \tag{4b}$$

$$\tilde{\theta}(0,x) = \tilde{\theta}_0(x), \tag{4c}$$

where $\tilde{\theta}(t,x)$, $s(t,x)$, and $u(t,x)$ are the system state, source term, and the control input respectively. $a > 0$, b, and c are the diffusivity, advectivity, and reactivity coefficients, respectively. Note that Assumption 1 holds.

Problem 1: Design an in-domain controller $u(t,x)$ for the system modeled by (4) such that, within a finite convergence time, we have $\lim\limits_{t \to T_1(\tilde{\theta})} \tilde{\theta}(t,x) = 0 \quad \forall x \in \Omega$.

In general, reaction-advection-diffusion PDEs, as described in (4), can be transformed into a form without the advection term using a gauge transformation (see [5]):

$$\hat{\theta}(t,x) = e^{\frac{bx}{2}} \tilde{\theta}(t,x). \tag{5}$$

Substituting (5) into (4) transforms the system into:

$$\hat{\theta}_t(t,x) = a\hat{\theta}_{xx}(t,x) + \bar{c}\hat{\theta}(t,x) + \hat{s}(t,x) + \hat{u}(t,x), \tag{6a}$$

$$\hat{\theta}(t,0) = 0, \quad \hat{\theta}(t,1) = 0, \tag{6b}$$

$$\hat{\theta}(0,x) = \hat{\theta}_0(x), \tag{6c}$$

here, $\bar{c} = c - \frac{1}{4}b^2$, $\hat{u}(t,x) = e^{\frac{bx}{2}} u(t,x)$, and $\hat{s}(t,x) = e^{\frac{bx}{2}} s(t,x)$. The transformation in (5) is invertible, rendering the system in (6) equivalent to the system in (4). Consequently, the primary objective for the system in (6) is $\hat{\theta}(t, x) \to 0$. Additionally, it should be noted that the heat transfer system described in (1) is also equivalent to the system in (6).

To address Problem 1, the following control input has been proposed:

$$u(t, x) = -s(t, x) + \gamma_1 \tilde{\theta}(t, x) + \gamma_2 \mathrm{sign}(\tilde{\theta}(t, x))|\tilde{\theta}(t, x)|^{\alpha}, \tag{7}$$

in which γ_1, γ_2, and α are design parameters.

Theorem 1: Consider the system described in (4) along with the proposed input control in (7). Note that Assumption 1 holds. Design $\gamma_1 < -(\frac{a}{4} + \bar{c}), \gamma_2 < 0$, and $\alpha \in (0, 1)$, then there exists a convergence time $T_1(\tilde{\theta})$ at which the system state is finite-time stabilized, satisfying the condition:

$$\lim_{t \to T_1(\tilde{\theta})} \tilde{\theta}(t, x) = 0, \quad \forall x \in \Omega, \tag{8}$$

where the convergence time is given by:

$$T_1(\tilde{\theta}) \leqslant \frac{1}{|\gamma_2|(1+\alpha)} \left(\int_0^1 e^{bx} \tilde{\theta}^2(t, x) \mathrm{d}x \right)^{\frac{1+\alpha}{2}}. \tag{9}$$

Proof: For the system in (6) consider the following Lyapunov function

$$V_1(\hat{\theta}) = \int_0^1 \hat{\theta}^2(t, x) \mathrm{d}x. \tag{10}$$

The time derivative of $V_1(\hat{\theta})$ is

$$\begin{aligned} \dot{V}_1(\hat{\theta}) =& 2 \int_0^1 \hat{\theta}(t, x) \hat{\theta}_t(t, x) \mathrm{d}x \\ =& 2a \int_0^1 \hat{\theta}(t, x) \hat{\theta}_{xx}(t, x) \mathrm{d}x + 2\bar{c} \int_0^1 \hat{\theta}^2(t, x) \mathrm{d}x \\ &+ 2 \int_0^1 \hat{\theta}(t, x) \hat{s}(t, x) \mathrm{d}x + 2 \int_0^1 \hat{\theta}(t, x) \hat{u}(t, x) \mathrm{d}x. \end{aligned} \tag{11}$$

By taking integration by parts from the first term and using Poincare inequality (See [5]) we have

$$\dot{V}_1(\hat{\theta}) \leqslant 2 \int_0^1 \hat{\theta}(t, x) \left((\frac{a}{4} + \bar{c}) \hat{\theta}(t, x) + \hat{s}(t, x) + \hat{u}(t, x) \right) \mathrm{d}x. \tag{12}$$

By considering (7) and (5) and substituting it in (12) we have

$$\begin{aligned} \dot{V}_1(\hat{\theta}) \leqslant& 2(\gamma_1 + \frac{a}{4} + \bar{c}) \int_0^1 \hat{\theta}^2(t, x) \mathrm{d}x \\ &+ 2\gamma_2 \int_0^1 |\hat{\theta}(t, x)|^{1+\alpha} \mathrm{d}x. \end{aligned} \tag{13}$$

Design $\gamma_1 < -(\frac{a}{4} + \bar{c})$, $\gamma_2 < 0$, and $\alpha \in (0, 1)$ then

$$\dot{V}_1(\hat{\theta}) \leqslant 2\gamma_2 \int_0^1 |\hat{\theta}(t, x)|^{1+\alpha} \mathrm{d}x = 2\gamma_2 \|\hat{\theta}(t, x)\|_{L_{1+\alpha}}^{1+\alpha}. \tag{14}$$

Note that the right hand side of (14) is the $1 + \alpha$ functional norm and $1+\alpha < 2$ then by using the norm property $\|\hat{\theta}\|_q \leqslant \|\hat{\theta}\|_p \leqslant n^{\frac{1}{p} - \frac{1}{q}} \|\hat{\theta}\|_q$ where $q > p > 0$ and $\hat{\theta} \in \mathbb{R}^n$, we have

$$\dot{V}_1(\hat{\theta}) \leqslant 2\gamma_2 \|\hat{\theta}(t, x)\|_{L_2}^{1+\alpha} = 2\gamma_2 V_1^{\frac{1+\alpha}{2}}(\hat{\theta}). \tag{15}$$

Since $\frac{1+\alpha}{2} \in (0, 1)$ and $\gamma_2 < 0$ then the Finite-Time stability Theorem [24] is satisfied and the convergence time is

$$T_1(\hat{\theta}) \leqslant \frac{1}{|\gamma_2|(1+\alpha)} \left(\int_0^1 \hat{\theta}^2(t, x) \mathrm{d}x \right)^{\frac{1+\alpha}{2}}, \tag{16}$$

where the Problem 1 is solved. ■

B. Boundary Control

Consider an inhomogeneous PDE as

$$\begin{aligned} \tilde{\theta}_t(t, x) =& a\tilde{\theta}_{xx}(t, x) + b\tilde{\theta}_x(t, x) + c\tilde{\theta}(t, x) + s(t, x), && \text{(17a)} \\ \tilde{\theta}(t, 0) =& 0, \quad \tilde{\theta}(t, 1) = u(t), && \text{(17b)} \\ \tilde{\theta}(0, x) =& \tilde{\theta}_0(x), && \text{(17c)} \end{aligned}$$

where $\tilde{\theta}(t, x), s(t, x)$, and $u(t)$ are the system state, source term, and the control input respectively. $a > 0$, b, and c are the diffusivity, advectivity, and reactivity coefficients, respectively. Note that Assumption 1 holds.

Problem 2: Design a boundary controller $u(t)$ for the system modeled by (17) such that, within a finite convergence time, we have $\lim_{t \to T_1(\tilde{\theta})} \tilde{\theta}(t, x) = 0 \quad \forall x \in \Omega$.

By taking into consideration of the gauge transform in (5), the system in (17) is transformed to the following system:

$$\begin{aligned} &\hat{\theta}_t(t, x) = a\hat{\theta}_{xx}(t, x) + \bar{c}\hat{\theta}(t, x) + \hat{s}(t, x), && \text{(18a)} \\ &\hat{\theta}(t, 0) = 0, \quad \hat{\theta}(t, 1) = e^{\frac{b}{2}} u(t) := \hat{u}(t), && \text{(18b)} \\ &\hat{\theta}(0, x) = \hat{\theta}_0(x). && \text{(18c)} \end{aligned}$$

Since the systems in (17) and (18) are equivalent, the objective for the system in (18) is $\hat{\theta}(t, x) \to 0$.

To address Problem 2, the following control input has been proposed:

$$\begin{aligned} u(t) =& \frac{b}{2} \tilde{\theta}(t, 1) + \tilde{\theta}_x(t, 1) \\ & - e^{\frac{-b}{2}} \sqrt{\left(\frac{b}{2} e^{\frac{b}{2}} \tilde{\theta}(t, 1) + e^{\frac{b}{2}} \tilde{\theta}_x(t, 1) \right)^2 + \frac{2}{a} \bar{f}_1(t)}, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \bar{f}_1(t) :=& \frac{e^{-1}}{2} \left(\frac{a}{2} + 2\bar{c} \right) \int_0^1 e^{x(b+1)} \tilde{\theta}^2(t, x) \mathrm{d}x \\ &+ e^{-1} \int_0^1 e^{x(b+1)} \tilde{\theta}(t, x) s(t, x) dx \\ &+ \frac{\gamma \cdot e^{-1}}{2} \left(\int_0^1 e^{x(b+1)} \tilde{\theta}^2(t, x) \mathrm{d}x \right)^{\alpha}, \end{aligned} \tag{20}$$

in which γ and α are design parameters.

Theorem 2: Consider the system described in (17) along with the proposed input control in (19). Note that Assumption 1 holds. Design $\gamma > 0$ and $\alpha \in (0, 1)$ then there exists a convergence time $T_1(\hat{\theta})$ at which the system state is finite-time stabilized, satisfying the condition:

$$\lim_{t \to T_1(\tilde{\theta})} \tilde{\theta}(t, x) = 0, \quad \forall x \in \Omega, \tag{21}$$

where the convergence time is given by:

$$T_1(\tilde{\theta}) \leqslant \frac{1}{\gamma(1-\alpha)} \left(\int_0^1 e^{x(b+1)} \tilde{\theta}^2(t,x) \right)^{1-\alpha}. \quad (22)$$

Proof: For the system in (18) consider the following Lyapunov function

$$V_2(\hat{\theta}) = \int_0^1 e^x \hat{\theta}^2(t,x) \mathrm{d}x. \quad (23)$$

The time derivative of $V_2(\hat{\theta})$ is

$$\begin{aligned} \dot{V}_2(\hat{\theta}) =& 2\int_0^1 e^x \hat{\theta}(t,x)\hat{\theta}_t(t,x)\mathrm{d}x \\ =& 2a\int_0^1 e^x \hat{\theta}(t,x)\hat{\theta}_{xx}(t,x)\mathrm{d}x + 2\bar{c}\int_0^1 e^x \hat{\theta}^2(t,x)\mathrm{d}x \\ &+ 2\int_0^1 e^x \hat{\theta}(t,x)\hat{s}(t,x)\mathrm{d}x. \end{aligned} \quad (24)$$

By taking integration by parts from the first term we have

$$\begin{aligned} \dot{V}_2(\hat{\theta}) =& -2a\int_0^1 e^x \hat{\theta}_x^2(t,x)\mathrm{d}x - 2a\int_0^1 e^x \hat{\theta}(t,x)\hat{\theta}_x(t,x)\mathrm{d}x \\ &+ 2a\left(e^x \hat{\theta}(t,x)\hat{\theta}_x(t,x)\Big|_0^1 \right) + 2\bar{c}\int_0^1 e^x \hat{\theta}^2(t,x)\mathrm{d}x \\ &+ 2\int_0^1 e^x \hat{\theta}(t,x)\hat{s}(t,x)\mathrm{d}x. \end{aligned} \quad (25)$$

By considering Poincare inequality for the first term and taking integration by parts from the second term and replacing the boundary conditions we have

$$\begin{aligned} \dot{V}_2(\hat{\theta}) \leqslant& -\frac{a}{2}\int_0^1 e^x \hat{\theta}^2(t,x)\mathrm{d}x + a\int_0^1 e^x \hat{\theta}^2(t,x)\mathrm{d}x \\ &- ae\hat{u}^2(t) + 2ae\hat{\theta}_x(t,1)\hat{u}(t) + 2\bar{c}\int_0^1 e^x \hat{\theta}^2(t,x)\mathrm{d}x \\ &+ 2\int_0^1 e^x \hat{\theta}(t,x)\hat{s}(t,x)\mathrm{d}x \\ \leqslant& \left(\frac{a}{2} + 2\bar{c}\right)\int_0^1 e^x \hat{\theta}^2(t,x)\mathrm{d}x - ae\hat{u}^2(t) \\ &+ 2ae\hat{\theta}_x(t,1)\hat{u}(t) + 2\int_0^1 e^x \hat{\theta}(t,x)\hat{s}(t,x)\mathrm{d}x \\ \leqslant& -2e\left(\frac{a}{2}\hat{u}^2(t) - a\hat{\theta}_x(t,1)\hat{u}(t) - e^{-1}\left(\frac{a}{4} + \bar{c}\right)V_2(\hat{\theta}) \right. \\ &\left. -e^{-1}\int_0^1 e^x \hat{\theta}(t,x)\hat{s}(t,x)\mathrm{d}x \right). \end{aligned} \quad (26)$$

By taking into consideration of $\bar{f}_1(t)$ in (20), we have

$$\begin{aligned} \dot{V}_2(\hat{\theta}) \leqslant& -2e\left(\frac{a}{2}\hat{u}^2(t) - a\hat{\theta}_x(t,1)\hat{u}(t) - \bar{f}_1(t) \right. \\ &\left. + \frac{\gamma \cdot e^{-1}}{2} V_2^{\alpha}(\hat{\theta}) \right). \end{aligned} \quad (27)$$

Design $\hat{u}(t)$ as

$$\hat{u}(t) = \hat{\theta}_x(t,1) - \sqrt{\hat{\theta}_x^2(t,1) + \frac{2}{a}\bar{f}_1(t)}, \quad (28)$$

in which by substituting (28) in (27) and reorganizing it, we have

$$\dot{V}_2(\hat{\theta}) \leqslant -\gamma V_2^{\alpha}(\hat{\theta}). \quad (29)$$

Choose $\alpha \in (0,1)$ and $\gamma > 0$ then the Finite-Time stability Theorem [24] is satisfied and the convergence time is

$$T_1(\hat{\theta}) \leqslant \frac{1}{\gamma(1-\alpha)} \left(\int_0^1 e^x \hat{\theta}^2(t,x)\mathrm{d}x \right)^{1-\alpha}, \quad (30)$$

where the Problem 2 is solved and by using (5), one may transform (28) to (19). ■

IV. Fixed-Time Stabilization

This section addresses inhomogeneous parabolic PDEs, focusing on designing controllers for fixed-time stabilization, independent of the system state. An in-domain controller is first developed for the system in (4), followed by a boundary controller for (17), both ensuring fixed-time stability.

A. In-domain Control

Consider the inhomogeneous system in (4) and note that Assumption 1 holds.

Problem 3: Design an in-domain controller $u(t,x)$ for the system modeled by (4) such that, within a finite convergence time, we have $\lim_{t\to T_2} \tilde{\theta}(t,x) = 0 \quad \forall x \in \Omega$.

To address Problem 3, the following control input has been proposed:

$$\begin{aligned} u(t,x) =& -s(t,x) + \gamma_1\tilde{\theta}(t,x) + \gamma_2 \mathrm{sign}(\tilde{\theta}(t,x))|\tilde{\theta}(t,x)|^{\alpha} \\ &+ \gamma_2 \mathrm{sign}(\tilde{\theta}(t,x))|\tilde{\theta}(t,x)|^{\beta}, \end{aligned} \quad (31)$$

in which γ_1, γ_2, β, and α are design parameters.

Theorem 3: Consider the system described in (4) along with the proposed input control in (31). Note that Assumption 1 holds. Design $\gamma_1 < -(\frac{a}{4} + \bar{c}), \gamma_2 < 0, \beta > 1$, and $\alpha \in (0,1)$, then there exists a convergence time T_2 at which the system state is fixed-time stabilized, satisfying the condition:

$$\lim_{t\to T_2} \tilde{\theta}(t,x) = 0, \quad \forall x \in \Omega, \quad (32)$$

where the convergence time is given by:

$$T_2 \leqslant \frac{1}{|\gamma_2|(1-\alpha)} + \frac{1}{|\gamma_2|(\beta-1)}. \quad (33)$$

Proof: For the system in (6) consider the following Lyapunov function

$$V_1(\hat{\theta}) = \int_0^1 \hat{\theta}^2(t,x)\mathrm{d}x. \quad (34)$$

By taking into consideration of (11) till (12) and considering (31) and (5) and substituting it in (12) we have

$$\begin{aligned} \dot{V}_1(\hat{\theta}) \leqslant& 2(\gamma_1 + \frac{a}{4} + \bar{c})\int_0^1 \hat{\theta}^2(t,x)\mathrm{d}x \\ &+ 2\gamma_2\int_0^1 |\hat{\theta}(t,x)|^{1+\alpha}\mathrm{d}x \\ &+ 2\gamma_2\int_0^1 |\hat{\theta}(t,x)|^{1+\beta}\mathrm{d}x. \end{aligned} \quad (35)$$

Consider that $\gamma_1 < -(\frac{a}{4}+\bar{c})$, $\gamma_2 < 0$, $\beta > 1$ and $\alpha \in (0,1)$ then

$$\dot{V}_1(\hat{\theta}) \leqslant 2\gamma_2 \left(\|\hat{\theta}(t,x)\|_{L_{1+\alpha}}^{1+\alpha} + \|\hat{\theta}(t,x)\|_{L_{1+\beta}}^{1+\beta} \right). \tag{36}$$

By using the norm property we have

$$\dot{V}_1(\hat{\theta}) \leqslant 2\gamma_2 \left(V_1^{\frac{1+\alpha}{2}}(\hat{\theta}) + V_1^{\frac{1+\beta}{2}}(\hat{\theta}) \right). \tag{37}$$

Since $\frac{1+\alpha}{2} \in (0,1)$, $\frac{1+\beta}{2} > 1$ and $\gamma_2 < 0$ then the Fixed-Time stability Theorem [20] is satisfied and the convergence time is

$$T_2 \leqslant \frac{1}{|\gamma_2|(1-\alpha)} + \frac{1}{|\gamma_2|(\beta-1)}, \tag{38}$$

where the Problem 3 is solved. ■

B. Boundary Control

Consider the inhomogeneous system in (17) and note that Assumption 1 holds.

Problem 4: Design a boundary controller $u(t)$ for the system modeled by (17) such that, within a finite convergence time, we have $\lim_{t\to T_2} \tilde{\theta}(t,x) = 0 \quad \forall x \in \Omega$.

To address Problem 4, the following control input has been proposed:

$$\begin{aligned} u(t,x) =& \frac{b}{2}\tilde{\theta}(t,1) + \tilde{\theta}_x(t,1) \\ & - e^{\frac{-b}{2}} \sqrt{\left(\frac{b}{2} e^{\frac{b}{2}} \tilde{\theta}(t,1) + e^{\frac{b}{2}} \tilde{\theta}_x(t,1) \right)^2 + \frac{2}{a}\bar{f}_2(t)}, \end{aligned} \tag{39}$$

where

$$\begin{aligned} \bar{f}_2(t) :=& \frac{e^{-1}}{2}\left(\frac{a}{2} + 2\bar{c}\right) \int_0^1 e^{x(b+1)} \tilde{\theta}^2(t,x)\mathrm{d}x \\ & + e^{-1} \int_0^1 e^{x(b+1)} \tilde{\theta}(t,x) s(t,x) dx \\ & + \frac{\gamma \cdot e^{-1}}{2} \left(\int_0^1 e^{x(b+1)} \tilde{\theta}^2(t,x)\mathrm{d}x \right)^{\alpha} \\ & + \frac{\gamma \cdot e^{-1}}{2} \left(\int_0^1 e^{x(b+1)} \tilde{\theta}^2(t,x)\mathrm{d}x \right)^{\beta}, \end{aligned} \tag{40}$$

in which γ, α, and β are design parameters.

Theorem 4: Consider the system described in (17) along with the proposed input control in (39). Note that Assumption 1 holds. Design $\gamma > 0$, $\alpha \in (0,1)$, and $\beta > 1$ then there exists a convergence time T_2 at which the system state is fixed-time stabilized, satisfying the condition:

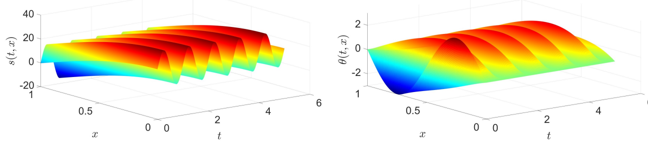
$$\lim_{t\to T_2} \tilde{\theta}(t,x) = 0, \quad \forall x \in \Omega, \tag{41}$$

where the convergence time is given by:

$$T_2 \leqslant \frac{1}{\gamma(1-\alpha)} + \frac{1}{\gamma(\beta-1)}. \tag{42}$$

Proof: For the system in (18) consider the following Lyapunov function

$$V_2(\hat{\theta}) = \int_0^1 e^x \hat{\theta}^2(t,x)\mathrm{d}x. \tag{43}$$



(a) The source term $s(x,t)$ evolution. (b) $\theta(t,x)$ evolution with $a_0 = 3$.

Fig. 1: Open-loop system

By taking into consideration of (24) till (26) and considering (40) and substituting it in (26) we have

$$\begin{aligned} \dot{V}_2(\hat{\theta}) \leqslant & -2e\left(\frac{a}{2}\hat{u}^2(t) - a\hat{\theta}_x(t,1)\hat{u}(t) - \bar{f}_2(t) \right. \\ & \left. + \frac{\gamma \cdot e^{-1}}{2} V_2^{\alpha}(\hat{\theta}) + \frac{\gamma \cdot e^{-1}}{2} V_2^{\beta}(\hat{\theta}) \right). \end{aligned} \tag{44}$$

Design $\hat{u}(t)$ as

$$\hat{u}(t) = \hat{\theta}_x(t,1) - \sqrt{\hat{\theta}_x^2(t,1) + \frac{2}{a}\bar{f}_2(t)}, \tag{45}$$

in which by substituting (45) in (44) and reorganizing it, we have

$$\dot{V}_2(\hat{\theta}) \leqslant -\gamma \left(V_2^{\alpha}(\hat{\theta}) + V_2^{\beta}(\hat{\theta}) \right). \tag{46}$$

Choose $\alpha \in (0,1)$, $\beta > 1$, and $\gamma > 0$ then the Fixed-Time stability Theorem [20] is satisfied and the convergence time is

$$T_2 \leqslant \frac{1}{\gamma(1-\alpha)} + \frac{1}{\gamma(\beta-1)}, \tag{47}$$

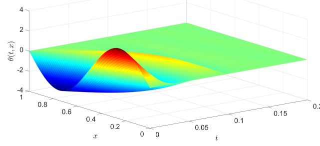
where the Problem 4 is solved and by using (5), one may transform (45) to (39). ■

V. Simulation

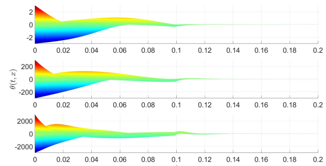
In this section, we discretize the systems in (4) and (17) with the proposed finite-time in-domain controller (7) and boundary controller (19), as well as the fixed-time in-domain controller (31) and boundary controller (39), following [4]. The parameters are $a = 1$, $b = 2$, $c = -5$, and the initial condition is $\tilde{\theta}_0(x) = a_0 \sin(2\pi x)$ with amplitudes $a_0 = 3$, 300, and 3000. The source term is $s(t,x) = 15\sin(2\pi t) + 22\cos(\pi x/2)$. Fig. 1a shows the source term evolution, and Fig. 1b illustrates the open-loop system behavior without control.

A. Finite-time Stabilization

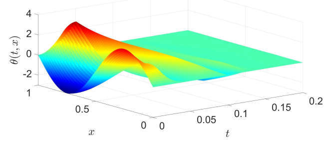
The parameters are set as $\gamma_1 = -5.75$, $\gamma_2 = -1$, $\gamma = 1$, and $\alpha = 0.5$. Fig. 2a shows the system state $\theta(t,x)$ under finite-time stable in-domain control for $a_0 = 3$, while Fig. 2b presents the convergence times for initial conditions with $\tilde{\theta}_0(x)$ amplitudes of 3, 300, and 3000 (top to bottom) for $x \in [0,1]$, indicating a dependence on initial conditions. Similarly, Fig. 2c and Fig. 2d illustrate the system state and convergence times under finite-time stable boundary control, also showing convergence time depends on initial conditions.



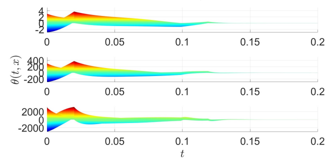
(a) In-domain control.



(b) Side view for $a_0 = 3, 300$, and 3000.

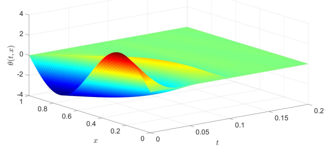


(c) Boundary control.

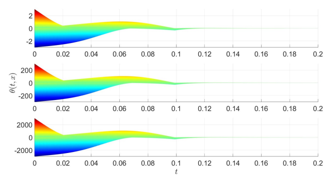


(d) Side view for $a_0 = 3, 300$, and 3000.

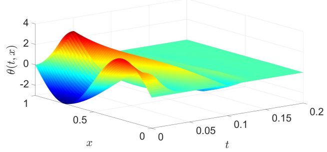
Fig. 2: $\theta(t, x)$ evolution with finite-time control design.



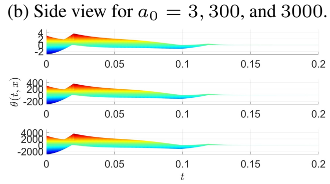
(a) In-domain control.



(b) Side view for $a_0 = 3, 300$, and 3000.



(c) Boundary control.



(d) Side view for $a_0 = 3, 300$, and 3000.

Fig. 3: $\theta(t, x)$ evolution with fixed-time control design.

B. Fixed-time Stabilization

The design parameters are set as $\gamma_1 = -5.75$, $\gamma_2 = -1$, $\gamma = 1$, $\alpha = 0.5$, and $\beta = 1.5$. Fig. 3a depicts the system state $\theta(t, x)$ under fixed-time stable in-domain control with $a_0 = 3$, while Fig. 3b shows convergence times for initial conditions with $\tilde{\theta}_0(x)$ amplitudes of 3, 300, and 3000 (top to bottom) for all $x \in [0, 1]$. As seen, convergence time is independent of initial conditions. Similarly, Fig. 3c and Fig. 3d illustrate the system state and convergence times, respectively, under fixed-time stable boundary control, demonstrating the same independence from initial conditions.

VI. Conclusion and Future works

This paper introduced innovative finite- and fixed-time stabilization techniques for inhomogeneous parabolic PDE systems, focusing on domain and boundary control strategies. We first designed controllers for finite-time stabilization, then addressed fixed-time stabilization, ensuring convergence time is independent of system states.

Future work will focus on developing finite- and fixed-time estimators for PDE-based systems and applying the stabilization methods from this paper to dynamic average consensus and encirclement in multi-agent systems.

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